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Response of an Inflatable Offshore Platform to Surface Wave Excitations

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The response to the ocean wave excitations of a neutrally buoyant inflatable offshore platform consisting of an array of three tapered inflatable legs attached to a central head, connected to a surface float by a cable is investigated. A Lagrangian formulation of the dynamics of the system is presented where the float and the central head are allowed to move vertically and the rotations of the array as well as the flexural displacements of the legs are superposed on this motion. The non-linear coupled system with infinite degrees of freedom is truncated by considering the first two terms in the admissible function expansion of each flexural displacement. This results in a set of ordinary differential equations which are converted to algebraic equations using Ritz's averaging technique. Representative plots showing the effect of system parameters on the platform response are presented.

Nomenclature

A_{ij},B_{ij}	= coefficients in the eigenvalue expansions of v_i and w_i, respectively; Eq. (4a)
a,a_b,a_h	= added inertia of each leg, the buoy, and head respectively
a_{ij},b_{ij}	= non-dimensionalized A_{ij} and B_{ij}, respectively
C_d,C_{db},C_{dh}	= drag coefficients of a leg, buoy, and head, respectively
C_m,C_{mb},C_{mh}	= added inertia coefficients of a leg, buoy, and head, respectively
C_θ,C_γ,C_ψ	= hydrodynamic moment coefficients
c	= equivalent stiffness due to buoyancy
d_0	= root diameter of each leg
E	= Young's modulus
F_H	= total hydrodynamic force on a body
$\bar{F}_b,\bar{F}_h,\bar{F}_i$	= hydrodynamic forces on the buoy, head, and ith leg, respectively
f,f_b,f_h	= coefficients of forcing functions, Eq. (14)
g	= acceleration due to gravity
H	= depth of the central head below the water surface, $Hc/g(m_b+a_b)$
h	= wall thickness of leg
I_0	= moment of inertia of the cross section of a leg at the root
I_i	$=2\pi(i-1)/3$; $i=1,2,3$
I_{xx},I_{yy},I_{zz}	= mass moments of inertia of the central head about x, y, z axes, respectively
I_{xxa},I_{yya},I_{zza}	= mass moments of inertia of the central head including added inertia effects
$\bar{I}_x,\bar{I}_y,\bar{I}_z$	= non-dimensionalized I_{xxa},I_{yya},I_{zza}, respectively; Eq. (14)
$\bar{i},\bar{j},\bar{k}$	= unit vectors along x, y, z axes, respectively
$\bar{i}_0,\bar{j}_0,\bar{k}_0$	= unit vectors along x_0, y_0, z_0 axes, respectively
$J_0,J_j,\bar{J}_j,J_{jn},K_{jn}$	= integrals involving Φ_j and ϵ, Eq. (8b) and (9b)
k	= stiffness of the cable
L	= length of the leg
m,m_b,m_h	= mass of each leg, the buoy, the head, respectively
m_0	= mass of a uniform leg with diameter d_0
p	= frequency of the wave
Q_n	= generalized force corresponding to the generalized coordinate q_n
q_n	= generalized coordinate
q_{nc},q_{ns}	= cosine and sine components of q_n
$[R]$	= transformation matrix; Eq. (2b)
r_m	$=m/m_0$
r_{hl},r_{bl}	= inertia parameters, Eq. (14)
$\bar{r}_b,\bar{r}_h$	= position vectors of the centers of mass of the buoy and head, respectively
$\dot{\bar{r}}\rvert_b$	= velocity of a point in the body coordinate axes
S,S_b,S_h	= areas of cross section of the leg, buoy, and head, respectively
T	= kinetic energy of the system
T_a,T_b,T_h	= kinetic energy of the legs, buoy, and head, respectively
t	= time
U	= potential energy of the system
U_e	= flexural strain energy stored in the legs
$V_{\rm rel}$	= relative velocity of an element with respect to the water
$\bar{V}_i$	= relative velocity of an element on the ith leg with respect to the water
$V_{\rm in},V_{\rm out}$	= inplane and out-of-plane components of $\bar{V}_i$, respectively
v_i,w_i	= inplane and out-of-plane flexural displacements of an element of the ith leg
$\bar{W}_i$	$=\bar{V}_i/d_0$

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W_{in}, W_{out} = $V_{in}/d_0, V_{out}/d_0$, respectively
x,y,z = body coordinate axes
x_0,y_0,z_0 = inertia coordinate axes
z_b,z_h = vertical displacements of the buoy and head, respectively
z_{b0} = equilibrium vertical displacement of the buoy
z_w,z_{wh},z_{wi} = displacement of a water particle at the buoy, central head, and a point on the ith leg, due to the ocean waves, respectively
$\alpha,\alpha_b,\alpha_h,$
$\alpha_\gamma,\alpha_\theta,\alpha_\psi$ = hydrodynamic damping coefficients, Eq. (14)
β_i = $(2\pi L/\lambda)\cos I_i$
$\Phi_j(\xi)$ = eigenfunctions of a cantilever, Eq. (4c)
Ω = frequency characteristic of individual legs, Eq. (14)
Ω_b = $(k/c)^{1/2}$
γ,ψ,θ = angles defining orientation of the array
δ_0 = phase difference between the crest of the wave and the vertical axis of symmetry of the system
δ_{mn} = Kronecker delta
ϵ = taper parameter (root diam − tip diam)/(root diam)
η_b,η_h = dimensionless vertical displacements of the buoy and head, respectively
$\eta_w,\eta_{wh},\eta_{wi}$ = $z_w/d_0, z_{wh}/d_0, z_{wi}/d_0$, respectively
λ = wave length
μ_j = jth eigenvalue of a cantilever
ξ = dimensionless distance from the root of the leg
ρ = density of water
σ = measure of energy dissipation in a viscoelastic material
σ_j = $(\cosh\mu_j + \cos\mu_j)/(\sinh\mu_j + \sin\mu_j)$
τ = dimensionless time, Eq. (14)
ψ_i = $\psi + I_i$
ω = dimensionless frequency of the wave
$\bar{\omega}$ = angular velocity, $\omega_x\bar{i} + \omega_x\bar{j} + \beta_z\bar{k}$

Dots and primes indicate differentiation with respect to t and τ, respectively.

I. Introduction

NEUTRALLY buoyant inflatable platforms have been proposed for a variety of applications. These platforms, which can be stabilized at any depth due to their neutral buoyancy, can serve as stations for equipment and instrumentation involved in oceanography survey, offshore exploration of minerals and oil, processing of resources, marine biological study, drilling operations, etc. If anchored or provided with magnetic commpasses, the system can be used as a navigation reference. When equipped with sensitive hydrophones, it can be employed as a submarine detection device. The platforms can have various possible configurations; but only the one that is quite promising will be considered here. It consists of an array of three tapered inflatable legs attached to a central head carrying a pump which pressurizes the tubes with water. The platform is connected to a surface float through an elastic cable which tends to minimize the wave disturbances. The surface float with its electronics serves as a telemetry link between the platform and a mother ship or a circling airplane where processing of information is conducted. As under normal operating conditions, the platform will be subjected to the ocean waves, the knowledge of the corresponding response is essential for its design.

Although general dynamical analyses of the buoy-cable-array assembly are indeed few, there is a vast body of literature dealing with individual constituents. The state-of-the-art in inflatable shell research, which has attracted considerable interest, is summarized by Leonard.[1] Quite relevant is the work of Kornecki[2] who has shown that, for the beam bending mode of shells without internal pressure, the Goldenveizer shell equations[3] reduce to an equation very similar to the one for transverse vibrations of a beam with rotary inertia included. The study of tapered beams with different boundary conditions has interested researchers for a long time. The first three natural frequencies and corresponding modes of cantilever beams for numerous different tapers were presented by Housner and Keightley.[4] Gaines and Volterra[5] derived approximate formulas for upper and lower values of the three lowest natural frequencies of cantilever bars of variable cross section. It must be emphasized that all these investigations pertain to either solid or hollow but empty tapered cantilevers. The free and forced vibrations of a neutrally buoyant inflated viscoelastic tapered cantilever in the presence of hydrodynamic forces was studied by Poon and Modi[6] in which they calculated its natural frequencies and the response to sinusoidal excitations. As regards the dynamics of the buoy-cable-array assembly, an analysis was presented by Modi and Misra.[7] However, the array was formed by uniform legs and the rotation of the central head and the legs was ignored. Furthermore, the vertical axis of the system was assumed to lie at the crest of a standing wave.

The objective of this paper is to study the general dynamics of the flexible platform formed by three inflated tapered tubes attached to a central head. The head is allowed to move vertically and the rotations of the array as well as the flexural displacements of the legs are superposed on this motion. The Lagrangian formulation of the problem is carried out accounting for the hydrodynamic forces. Subsequently, the response to a sinusoidal standing ocean wave is obtained by using Ritz's averaging technique. Attempts are made to determine the effects of taper ratio and other parameters on the tip displacements of the tubes.

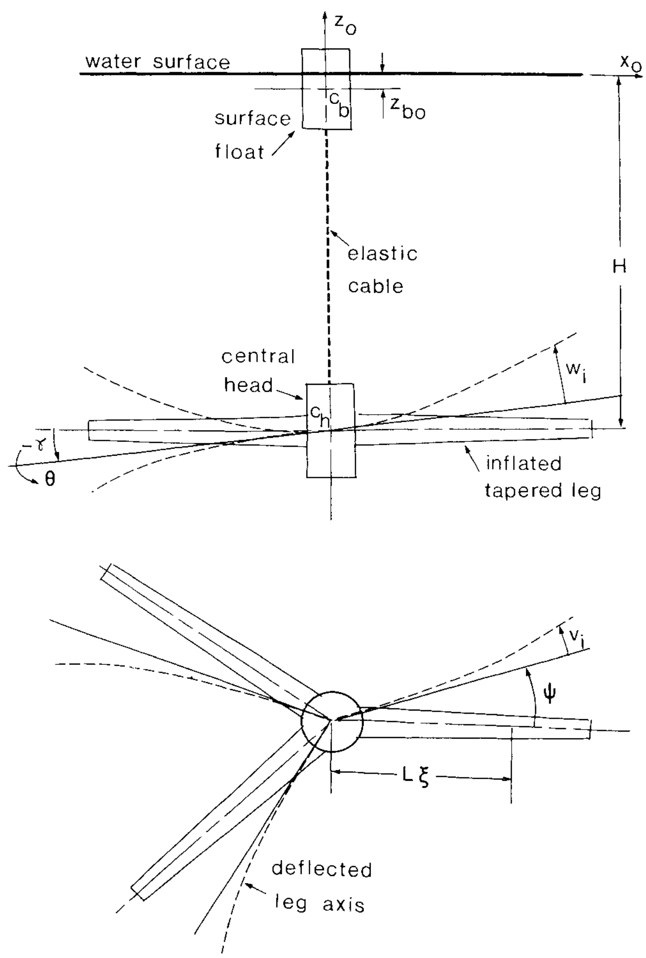

Fig. 1 Geometry of motion of the flexible platform.

II. Formulation of the Problem

The platform consists of a central head of mass m_h and three identical inflated tubes characterized by root diameter d_0, taper parameter ϵ, thickness h, and length L. It is connected to a surface float by an elastic cable which may be represented by a simple spring of stretching stiffness k, since its stiffness in any other dierction is comparatively much larger. It is assumed that the surface float is not drifting horizontally (Fig. 1).

The inertial co-ordinate axes x_0, y_0, z_0 are located such that the z_0 axis coincides with the vertical axis of symmetry of the system and the x_0 axis lying on the nominal water surface is parallel to one of the legs. The y_0 axis is oriented to complete a right-handed orthogonal system. The coordinates of the equilibrium positions of the centers of mass of the surface float and the central head are (0, 0, z_{b0}) and ((0, 0, $-H$), respectively, where H is the depth below the water surface where the platform is stabilized. At any instant t, let z_b and z_h be the corresponding vertical displacements from the equilibrium configuration; i.e.,

$$\bar{r}_b = (z_{b0} + z_b)\bar{k}_0 \tag{1a}$$

$$\bar{r}_h = (-H + z_h)\bar{k}_0 \tag{1b}$$

where $\bar{i}_0$, $\bar{j}_0$, and $\bar{k}_0$ are the unit vectors along the inertial coordinate axes. The orientation of the platform is specified by a sequence of modified Eulerian rotations γ, θ, and ψ. The relation between the inertial coordinates x_0, y_0, z_0 and array body coordinates x, y, z is

$$(r_0) = [R](r) + (r_h) \tag{2a}$$

where the transformation matrix $[R]$ is given by

$$[R] = \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2b}$$

The flexural displacements of a point on the ith leg at a distance $\xi L (0 \le \xi \le 1)$ from the root can be resolved into two components: v_i in the plane of the array and w_i perpendicular to it. Hence, the body coordinates of the above point are components of

$$\bar{r} = (\xi L \cos I_i - v_i \sin I_i)\bar{i} + (\xi L \sin I_i + v_i \cos I_i)\bar{j} + w_i \bar{k} \tag{3a}$$

where

$$I_i = 2\pi(i-1)/3 \tag{3b}$$

and $\bar{i}$, $\bar{j}$, $\bar{k}$ are unit vectors along the x, y, z directions, respectively.

The deformations v_i and w_i may be expanded in series forms

$$v_i = \sum_{j=1}^{\infty} A_{ij}\phi_j(\xi), \qquad w_i = \sum_{j=1}^{\infty} B_{ij}\Phi_j(\xi) \tag{4a}$$

where $\Phi_j(\xi)$, $j = 1, 2, \ldots \infty$ are a set of orthonormal functions satisfying the geometric as well as dynamic boundary conditions given by

$$\Phi_j(0) = \frac{d\Phi_j}{d\xi}(0) = \frac{d^2\Phi_j}{d\xi^2}(1) = \frac{d^3\Phi_j}{d\xi^3}(1) = 0 \tag{4b}$$

For a given accuracy, a smaller number of terms is needed if $\Phi_j(\xi)$ are the eigenfunctions of the tapered leg under consideration; however, the advantage is affected by the fact that the algebra involved is quite complicated. Hence, the modes of a uniform cylindrical cantilever,[8] which satisfy Eq. (4b) are chosen; i.e.,

$$\Phi_j(\xi) = (\cosh\mu_j\xi - \cos\mu_j\xi) - \sigma_j(\sinh\mu_j\xi - \sin\mu_j\xi) \tag{4c}$$

where μ_j are the roots of the equation

$$1 + \cosh\mu \cos\mu = 0 \tag{4d}$$

and

$$\sigma_j = (\cosh\mu_j + \cos\mu_j)/(\sinh\mu_j + \sin\mu_j) \tag{4e}$$

The kinetic energy T of the system is given by

$$T = (m_b/2)\dot{z}_b^2 + (1/2)[m_h\dot{z}_h^2 + I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2]$$

$$+ (1/2)\int_{\text{legs}} (\dot{\bar{r}}_h + \dot{\bar{r}}|_b + \bar{\omega}x\bar{r}) \cdot (\dot{\bar{r}}|_b + \bar{\omega}x\bar{r})\, dm \tag{5}$$

where I_{xx}, I_{yy} and I_{zz} are the moments of inertia of the central head about the x, y, and z axis, respectively (assumed to be principal axes), $\dot{\bar{r}}|_b$ is the velocity of the point in the body coordinate axes and $\bar{\omega}$ is the angular velocity of the array. Clearly,

$$\dot{\bar{r}}|_b = -\dot{v}_i \sin I_i \bar{i} + \dot{v}_i \cos I_i \bar{j} + \dot{w}_i \bar{k} \tag{6a}$$

and

$$\bar{\omega} = \omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k} = (\dot{\gamma}\sin\psi\cos\theta + \dot{\theta}\cos\psi\bar{i}$$

$$+ (\dot{\gamma}\cos\psi\cos\theta - \dot{\theta}\sin\psi)\bar{j} + (-\dot{\gamma}\sin\theta + \dot{\psi})\bar{k} \tag{6b}$$

The mass dm of the element corresponds to the mass of the water inside the tube; the mass of the cylindrical membrane itself is negligible. Hence,

$$dm = m_0(1 - \epsilon\xi)^2 d\xi \tag{7a}$$

where m_0 is the mass of the water inside a uniform cylindrical tube with its diameter equal to the root diameter of the tapered leg. Integration of Eq. (7a) yields.

$$r_m = m/m_0 = 1 - \epsilon + (\epsilon^2/3) \tag{7b}$$

Substitution of Eq. (6) and (7) in Eq. (5) yields

$$T = \left(\frac{3m}{2}\right)\dot{z}_h^2 + m_0 \sum_{i=1}^{3} \left(\frac{J_0 L^2}{2}\right)\{(-\dot{\gamma}\cos\theta\cos\psi_i + \dot{\theta}\sin\psi_i)^2$$

$$+ (-\dot{\gamma}\sin\theta + \dot{\psi})^2\}$$

$$+ \sum_{j=1}^{\infty} \dot{z}_h J_j [\{\dot{A}_{ij}(\sin\gamma\sin\psi_i + \cos\gamma\sin\theta\cos\psi_i)$$

$$+ \dot{B}_{ij}\cos\gamma\cos\theta\} + A_{ij}\{\cos\gamma\cos\theta(\dot{\gamma}\cos\theta\sin\psi_i + \dot{\theta}\cos\psi_i)$$

$$- (-\dot{\gamma}\sin\theta + \dot{\psi})(-\sin\gamma\cos\psi_i + \cos\gamma\sin\theta\sin\psi_i)\}$$

$$-B_{ij}(\dot\gamma\sin\gamma\cos\theta+\dot\theta\cos\gamma\sin\theta)]+\sum_{j=1}^{\infty}L\tilde J_j[\{\dot A_{ij}(-\dot\gamma\sin\theta+\dot\psi)+\dot B_{ij}(-\dot\gamma\cos\theta\cos\psi_i+\dot\theta\sin\psi_i)\}$$

$$+A_{ij}\{\tfrac{1}{2}(\dot\theta^2-\dot\gamma^2\cos^2\theta)\sin 2\psi_i-\dot\gamma\dot\theta\cos\theta\cos 2\psi_i\}-B_{ij}(\dot\gamma\cos\theta\sin\psi_i+\dot\theta\cos\psi_i)(-\dot\gamma\sin\theta+\dot\psi)]$$

$$+(\tfrac{1}{2})\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}J_{jn}[(\dot A_{ij}\dot A_{in}+\dot B_{ij}\dot B_{in})+2(A_{ij}\dot B_{in}-\dot A_{ij}B_{in})(\dot\gamma\cos\theta\sin\psi_i+\dot\theta\cos\psi_i)+A_{ij}A_{in}\{(\dot\gamma\cos\theta\sin\psi_i+\dot\theta\cos\psi_i)^2$$

$$+(-\dot\gamma\sin\theta+\dot\psi)^2\}+B_{ij}B_{in}(\dot\gamma^2\cos^2\theta+\dot\theta^2)+2A_{ij}B_{in}(-\dot\gamma\sin\theta+\dot\psi)(-\dot\gamma\cos\theta\cos\psi_i+\dot\theta\sin\psi_i)]+(m_b/2)\dot z_b^2$$

$$+(\tfrac{1}{2})[m_h\dot z_h^2+I_{xx}(\dot\gamma\sin\psi\cos\theta+\dot\theta\cos\psi)^2+I_{yy}(\dot\gamma\cos\psi\cos\theta-\dot\theta\sin\psi)^2+I_{zz}(-\dot\gamma\sin\theta+\dot\psi)^2] \quad (8a)$$

where

$$\psi_i=\psi+I_i$$

$$J_0=\int_0^1\xi^2(1-\epsilon\xi)^2\mathrm{d}\xi=(\tfrac{1}{3})-(\epsilon/2)+(\epsilon^2/5)$$

$$J_j=\int_0^1\Phi_j(1-\epsilon\xi)^2\mathrm{d}\xi=(2\sigma_j/\mu_j)[1-(2/\sigma_j)(\epsilon/\mu_j)+(-1)^{j+1}2(\epsilon/\mu_j)^2]$$

$$\tilde J_j=\int_0^1\xi\Phi_j(1-\epsilon\xi)^2\mathrm{d}\xi=(2/\mu_j^2)[1+(-1)^{j+1}\{-4\sigma_j(\epsilon/\mu_j)+6(\sigma_j\mu_j-1)(\epsilon/\mu_j)^2\}]$$

and

$$J_{jn}=\int_0^1\Phi_j\Phi_n(1-\epsilon\xi)^2\mathrm{d}\xi=\delta_{mn}-2\epsilon\int_0^1\xi\Phi_m\Phi_n d\xi+\epsilon^2\int_0^1\xi^2\Phi_m\Phi_n d\xi \quad (8b)$$

One may note the added complexity due to nonzero taper parameter ϵ.

The strain energy U_e stored in the legs is given by

$$U_e=(\tfrac{1}{2})\sum_{i=1}^{3}EI_0\int_0^L(1-\epsilon\xi)^3\left[\left(\frac{\partial^2 v_i}{\partial(L\xi)^2}\right)^2+\left(\frac{\partial^2 w_i}{\partial(L\xi)^2}\right)^2\right]\mathrm{d}(L\xi)=(EI_0/2L^3)\sum_{i=1}^{3}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}K_{jn}(A_{ij}A_{in}+B_{ij}B_{in}) \quad (9a)$$

with

$$K_{jn}=\int_0^1(1-\epsilon\xi)^3\Phi_j''\Phi_n''\mathrm{d}\xi \quad (9b)$$

The total potential energy of the system is the sum of the strain energy in legs, the energy stored in the cable, and that arising due to the buoyancy of the surface float; i.e.,

$$U=(EI_0/2L^3)\sum_{i=1}^{3}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}K_{jn}(A_{ij}A_{in}+B_{ij}B_{in})+(k/2)(z_b-z_h)^2+(c/2)(z_b-z_w)^2 \quad (9c)$$

where z_w is the surface wave displacement and $c=(\rho g)$ (area of the cross section of the buoy). The equations of motion are obtained from

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial\dot q_n}\right]-\frac{\partial T}{\partial q_n}+\frac{\partial U}{\partial q_n}=Q_n \quad (10)$$

$$q_n=z_b,z_h,\gamma,\theta,\psi,A_{ij},B_{ij};\qquad i=1,2,3;\qquad j=1,2,\ldots\infty$$

where Q_n are the generalized forces arising due to the hydrodynamic drag and added inertia.

A. Evaluation of Generalized Forces

The hydrodynamic force acting on an element may be resolved into two components,[9] one proportional to its relative acceleration and the other varying as the square of its velocity with respect to the fluid; i.e.,

$$F_h=-\left[C_m\mathrm{d}m\frac{\mathrm{d}V_{\mathrm{rel}}}{\mathrm{d}t}+\left(\frac{\rho}{2}\right)C_d\mathrm{d}S\,V_{\mathrm{rel}}|V_{\mathrm{rel}}|\right] \quad (11)$$

where C_m and C_d are the coefficients of added inertia and drag, respectively and $\mathrm{d}S$ is the projected area of the element normal to the relative velocity, V_{rel}. Hence, the hydrodynamic force acting on the surface float may be written as

$$\bar F_b=[-m_bC_{mb}(\ddot z_b-\ddot z_w)-(\rho/2)C_{db}S_b(\dot z_b-\dot z_w)|\dot z_b-\dot z_w|]\bar k_0 \quad (12a)$$

where C_{mb}, C_{db}, and S_b are the added inertia and drag coefficients and area of the cross section of the buoy, respectively, while z_w is the displacement of a water particle at the center of mass of the buoy. The hydrodynamic effect on the central head are taken into account by a vertical force $\bar{F}_h$ and a moment $\bar{M}_h$ given by

$$\bar{F}_h = [-m_h C_{mh}(\ddot{z}_h - \ddot{z}_{wh}) - (\rho/2) C_{dh} S_h (\dot{z}_h - \dot{z}_{wh}) |\dot{z}_h - \dot{z}_{wh}| \bar{k}_0] \tag{12b}$$

$$\bar{M}_h = [-(A_{\theta\theta}\ddot{\theta}\bar{i}_0 + A_{\gamma\gamma}\ddot{\gamma}\bar{j}_0 + A_{\psi\psi}\ddot{\psi}\bar{k}_0) - (\rho/2) S_h d_h^3 (C_\theta \dot{\theta}|\dot{\theta}|\bar{i}_0 + C_\gamma \dot{\gamma}|\dot{\gamma}|\bar{j}_0 + C_\psi \dot{\psi}|\dot{\psi}|\bar{k}_0)] \tag{12c}$$

Here, C_{mh}, $A_{\theta\theta}$, $A_{\gamma\gamma}$, $A_{\psi\psi}$ characterize the added inertia effects, whileC_{dh}, C_θ, C_γ, C_ψ determine the hydrodynamic damping. The displacement of a water particle at the center of mass of the central head is denoted by z_{wh}. The hydrodynamic force acting on an element $Ld\xi$ of the ith leg is given by

$$\mathrm{d}\bar{F}_i = [-(\rho/2) C_d L d_0 (1 - k\xi) \bar{V}_i |\bar{V}_i| - C_m \rho A_0 L (1 - \epsilon\xi)^2 \dot{\bar{V}}_i] \mathrm{d}\xi \tag{12d}$$

where C_d and C_m are drag and added inertia coefficient of a cylinder, respectively, A_0 the area of cross section of the leg at the root and $\bar{V}_i$ the relative velocity of the element with respect to the water. Strictly speaking, the component of $\bar{V}_i$ normal to the element should be considered. However, for small displacements, the error involved is negligible.

The generalized forces are evaluated from Eqs. (12) by using the principle of virtual work.

B. Linearized Equations of Motion

Substitution of Eqs. (8) and (9) in Eq. (10) yields a system of rather lengthy coupled nonlinear ordinary differential equations which is not tractable. Hence, some simplifications must be made. The displacements are assumed to be small so that the second and higher order terms may be neglected compared to the first order terms. However, in the algebraic expressions describing the velocity square damping, the second order terms are retained while the subsequent higher order terms are ignored. With these approximations, the equations of motion are:

$$(m_b + a_b)\ddot{z}_b + (c+k) z_b - k z_h + \tfrac{1}{2}\rho C_{db} S_b (\dot{z}_b - \dot{z}_w)|\dot{z}_b - \dot{z}_w| = c z_w + a_b \ddot{z}_w \tag{13a}$$

$$[(m_h + a_h) + 3m(1+C_m)]\ddot{z}_h + m_0(1+C_m)\sum_{i=1}^{3}\sum_{j=1}^{\infty} J_j \ddot{B}_{ij} + k(z_h - z_b) + \tfrac{1}{2}\rho C_{dh} S_h (\dot{z}_h - \dot{z}_{wh})|\dot{z}_h - \dot{z}_{wh}|$$
$$+ \tfrac{1}{2}\rho C_d L d_0 \sum_{i=1}^{3}\int_0^1 |\bar{V}_i| V_{\text{out}} (1-\epsilon\xi)\mathrm{d}\xi = a_h \ddot{z}_{wh} + m_0 C_m \sum_{i=1}^{3}\int_0^1 \ddot{z}_{wi}(1-\epsilon\xi)^2 \mathrm{d}\xi \tag{13b}$$

$$m_0 L^2 (1+C_m)\left[\left(\frac{3J_0}{2}\right)\ddot{\gamma} - \sum_{i=1}^{3}\sum_{j=1}^{\infty}\left(\frac{\ddot{B}_{ij}}{L}\right)\bar{J}_j \cos I_i\right] + I_{yya}\ddot{\gamma} + \tfrac{1}{2}\rho S_h d_h^3 C_\gamma \dot{\gamma}|\dot{\gamma}|$$
$$+ \tfrac{1}{2}\rho C_d L^2 d_0 \sum_{i=1}^{3}\int_0^1 |\bar{V}_i|(-\xi \cos I_i) V_{\text{out}}(1-\epsilon\xi)\mathrm{d}\xi = -m_0 C_m L \sum_{i=1}^{3} \cos I_i \int_0^1 \ddot{z}_{wi}\xi(1-\epsilon\xi)^2 \mathrm{d}\xi \tag{13c}$$

$$m_0 L^2 (1+C_m)\left[\left(\frac{3J_0}{2}\right)\ddot{\theta} + \sum_{i=1}^{3}\sum_{j=1}^{\infty}\left(\frac{\ddot{B}_{ij}}{L}\right)\bar{J}_j \sin I_i\right] + I_{xxa}\ddot{\theta} + \tfrac{1}{2}\rho S_h d_h^3 C_\theta \dot{\theta}|\dot{\theta}|$$
$$+ \tfrac{1}{2}\rho C_d L^2 d_0 \sum_{i=1}^{3}\int_0^1 |\bar{V}_i|(\xi \sin I_i) V_{\text{out}}(1-\epsilon\xi)\mathrm{d}\xi = m_0 C_m L \sum_{i=1}^{3} \sin I_i \int_0^1 \ddot{z}_{wi}\xi(1-\epsilon\xi)^2 \mathrm{d}\xi \tag{13d}$$

$$m_0(1+C_m)\left[J_j \ddot{z}_h + \bar{J}_j L(\ddot{\theta}\sin I_i - \ddot{\gamma}\cos I_i) + \sum_{n=1}^{\infty} J_{nj}\ddot{B}_{in}\right] + \left(\frac{EI_0}{L^3}\right)\sum_{n=1}^{\infty} K_{nj} B_{in} + \tfrac{1}{2}\rho C_d L d_0 \int_0^1 \Phi_j |\bar{V}_i| V_{\text{out}}(1-\epsilon\xi)\mathrm{d}\xi$$
$$= m_0 C_m \int_0^1 \Phi_j \ddot{z}_{wi}(1-\epsilon\xi)^2 \mathrm{d}\xi \quad i=1,2,3; j=1,2,\ldots\infty \tag{13e}$$

$$m_0 L^2 (1+C_m)\left[3J_0\ddot{\psi} + \sum_{i=1}^{3}\sum_{j=1}^{\infty}\left(\frac{\ddot{A}_{ij}}{L}\right)\bar{J}_j\right] + I_{zza}\ddot{\psi} + \tfrac{1}{2}\rho S_h d_h^3 \dot{\psi}|\dot{\psi}| + \tfrac{1}{2}\rho C_d L^2 d_0 \sum_{i=1}^{3}\int_0^1 |\bar{V}_i| V_{\text{in}}\xi(1-\epsilon\xi)\mathrm{d}\xi = 0 \tag{13f}$$

$$m_0(1+C_m)\left[\bar{J}_j L\ddot{\psi} + \sum_{n=1}^{\infty} J_{nj}\ddot{A}_{in}\right] + \left(\frac{EI_0}{L^3}\right)\sum_{n=1}^{\infty} K_{nj} A_{\text{in}} + \tfrac{1}{2}\rho C_d L d_0 \int_0^1 \Phi_j |\bar{V}_i| V_{\text{in}}(1-\epsilon\xi)\mathrm{d}\xi = 0 \quad i=1,2,3; j=1,2,\ldots,\infty \tag{13g}$$

where I_{xxa}, I_{yya}, I_{zza} are apparent moments of inertia of the central head,

$$V_{\text{in}} = \dot{v}_i + \xi L\dot{\psi} \tag{13h}$$

$$V_{\text{out}} = \dot{z}_h + \xi L(\dot{\theta}\sin I_i - \dot{\gamma}\cos I_i) + \dot{w}_i - \dot{z}_{wi} \tag{13i}$$

$$|\bar{V}_i| = \text{magnitude of the velocity of an element of the } i\text{th leg relative to the water} = [V_{\text{in}}^2 + V_{\text{out}}^2]^{1/2} \tag{13j}$$

and z_{wi} is the wave displacement at the element. Defining

$$\eta_b = z_b/d_0, \quad \eta_h = z_h/d_0, \quad a_{ij} = A_{ij}/d_0, \quad b_{ij} = B_{ij}/d_0, \quad \eta_w = z_w/d_0, \quad \eta_{wh} = z_{wh}/d_0, \quad \eta_{wi} = z_{wi}/d_0, \quad \tau = t[c/(m_b + a_b)]^{1/2}$$

Equation (13) may be nondimensonalized to obtain

$$\eta_b'' + (1 + \Omega_b^2)\eta_b - \Omega_b^2\eta_h + \alpha_b(\eta_b' - \eta_w')|\eta_b' - \eta_w'| = \eta_w + f_b\eta_w'' \tag{14a}$$

$$(r_{hl} + 3r_m)\eta_h'' + \sum_{i=1}^{3}\sum_{j=1}^{\infty} J_j b_{ij}'' + \Omega_b^2 r_{bl}(\eta_h - \eta_b) + \alpha_h(\eta_h' - \eta_{wh}')|\eta_h' - \eta_{wh}'| + \alpha\sum_{i=1}^{3}\int_0^1 |\bar{W}_i| W_{\text{out}}(1 - \epsilon\xi)d\xi$$
$$= f_h\eta_{wh}'' + f\sum_{i=1}^{3}\int_0^1 \eta_{wi}''(1 - \epsilon\xi)^2 d\xi \tag{14b}$$

$$\left(\frac{3}{2}J_0 + \bar{I}_y\right)\gamma'' - \left(\frac{d_0}{L}\right)\sum_{i=1}^{3}\sum_{j=1}^{\infty}\bar{J}_j b_{ij}''\cos I_i + \alpha_\gamma\gamma'|\gamma'| + \alpha\left(\frac{d_0}{L}\right)\sum_{i=1}^{3}\int_0^1 |\bar{W}_i| W_{\text{out}}(1 - \epsilon\xi)(-\xi\cos I_i)d\xi$$
$$= -f\left(\frac{d_0}{L}\right)\sum_{i=1}^{3}\cos I_i\int_0^1 \eta_{wi}''\xi(1 - \epsilon\xi)^2 d\xi \tag{14c}$$

$$\left(\frac{3}{2}J_0 + \bar{I}_x\right)\theta'' + \left(\frac{d_0}{L}\right)\sum_{i=1}^{3}\sum_{j=1}^{\infty}\bar{J}_j b_{ij}''\sin I_i + \alpha_\theta\theta'|\theta'| + \alpha\left(\frac{d_0}{L}\right)\sum_{i=1}^{3}\int_0^1 |\bar{W}_i| W_{\text{out}}(1 - \epsilon\xi)(\xi\sin I_i)d\xi$$
$$= f\left(\frac{d_0}{L}\right)\sum_{i=1}^{3}\sin I_i\int_0^1 \eta_{wi}''\xi(1 - \epsilon\xi)^2 d\xi \tag{14d}$$

$$J_j\eta_h'' + \bar{J}_j\left(\frac{L}{d_0}\right)(\theta''\sin I_i - \gamma''\cos I_i) + \sum_{n=1}^{\infty}[J_{nj}b_{in}'' + \Omega^2 K_{nj}b_{in}] + \alpha\int_0^1 \Phi_j|\bar{W}_i| W_{\text{out}}(1 - \epsilon\xi)d\xi = f\int_0^1 \Phi_j(1 - \epsilon\xi)^2\eta_{wi}''d\xi \tag{14e}$$

$$(3J_0 + \bar{I}_z)\psi'' + \left(\frac{d_0}{L}\right)\sum_{i=1}^{3}\sum_{j=1}^{\infty}\bar{J}_j a_{ij}'' + \alpha_\psi\psi'|\psi'| + \alpha\left(\frac{d_0}{L}\right)\sum_{i=1}^{3}\int_0^1 |\bar{W}_i| W_{\text{in}}(1 - \epsilon\xi)\xi d\xi = 0 \tag{14f}$$

$$\bar{J}_j\left(\frac{L}{d_0}\right)\psi'' + \sum_{n=1}^{\infty}[J_{nj}a_{in}'' + \Omega^2 k_{nj}a_{in}] + \alpha\int_0^1 \Phi_j|\bar{W}_i| W_{\text{in}}(1 - \epsilon\xi)d\xi = 0 \tag{14g}$$

where

$$\Omega_b^2 = k/c, \quad \Omega^2 = [EI_0/m_0(1 + C_m)L^3][(m_b + a_b)/c]$$

$$r_{bl} = (m_b + a_b)/m_0(1 + C_m), \quad r_{hl} = (m_h + a_h)/m_0(1 + C_m)$$

$$\bar{I}_x = I_{xxa}/m_0(1 + C_m)L^2, \quad \bar{I}_y = I_{yya}/m_0(1 + C_m)L^2$$

$$\bar{I}_z = I_{zza}/m_0(1 + C_m)L^2, \quad \alpha_b = (\rho/2)C_{db}S_b d_0/(m_b + a_b)$$

$$\alpha_h = (\rho/2)C_{dh}S_h d_0/m_0(1 + C_m), \quad \alpha = 2C_d/\pi(1 + C_m)$$

$$\alpha_\gamma = (\rho/2)S_h d_h^3 C_\gamma/m_0 L^2(1 + C_m), \quad \alpha_\theta = C_\theta(\alpha_\gamma/C_\gamma)$$

$$\alpha_\psi = C_\psi(\alpha_\gamma/C_\gamma), \quad f_b = a_b/(m_b + a_b)$$

$$f_h = a_h/m_0(1 + C_m), \quad f = C_m/(1 + C_m)$$

$$\bar{W}_i = \bar{V}_i/d_0, \quad W_{\text{in}} = V_{\text{in}}/d_0, \quad W_{\text{out}} = V_{\text{out}}/d_0$$

and prime denotes differentiation with respect to τ. In order to incorporate viscoelastic effects of the legs, the elastic modulus E is replaced by $E[1 + \sigma(\partial/\partial\tau)]$ where σ is a measure of energy dissipation. This results in the use of the operator $\Omega^2[1 + \sigma(\partial/\partial\tau)]$ instead of Ω^2 in Eq. (14).

C. Response of the System to Ocean Waves

The system, under normal operating conditions, will be subjected to the ocean waves. In the present analysis, its response to a sinusoidal plane standing wave with no variation in the direction of y_0 is determined. The motion forced by a more complex wave can be obtained by following an approximate analytical procedure similar to the one used here. It can be shown[10] that, for the standing wave considered, the wave displacement at the point having inertial coordinates x_0, y_0, z_0 is $\eta_0 \exp(-2\pi z_0/\lambda)\cos pt\cdot\cos 2\pi(x_0 - x_{00})/\lambda$ where η_0, p, and λ are the amplitude, frequency, and length of the wave, respectively, while x_{00} locates its crest. Hence,

$$\eta_w = \eta_0\cos\omega\tau\cos\delta_0 \tag{15a}$$

$$\eta_{wh} = \eta_0\exp(-2\pi H/\lambda)\cos\omega\tau\cos\delta_0 \tag{15b}$$

$$\eta_{wi} = \eta_0\exp(-2\pi H/\lambda)\cos\omega\tau\cos(\beta_i\xi - \delta_0) \tag{15c}$$

where

$$\omega = \text{dimensionless frequency} = p[(m_b + a_b)/c]^{1/2}$$

$$\delta_0 = (2\pi x_{00}/\lambda, \qquad \beta_i = (2\pi L/\lambda)\cos I_i$$

It may be pointed out that p and λ are not independent and are related by $p^2 = 2\pi g/\lambda$.

This forced motion, in general, will involve all the harmonics of ω; but for simplicity, only the fundamental term, which is usually the most important one, is considered. In order to account for system damping, both sine and cosine terms should be included in the solution. Hence,

$$\eta_b = \eta_{bc}\cos\omega\tau + \eta_{bs}\sin\omega\tau, \quad \eta_h = \eta_{hc}\cos\omega\tau + \eta_{hs}\sin\omega\tau$$

$$a_{ij} = a_{ijc}\cos\omega\tau + a_{ijs}\sin\omega\tau, \quad b_{ij} = b_{ijc}\cos\omega\tau + b_{ijs}\sin\omega\tau$$

$$q = q_c\cos\omega\tau + q_s\sin\omega\tau, \quad q = \gamma, \theta, \psi \tag{16}$$

Clearly,

$$W_{\text{in}} = \left[\xi L\psi_c + \sum_{j=1}^{\infty} A_{ijc}\Phi_j(\xi)\right](-\omega\sin\omega\tau) + \left[\xi L\psi_s + \sum_{j=1}^{\infty} A_{ijs}\Phi_j(\xi)\right](\omega\cos\omega\tau)$$
$$= \omega[-W_{\text{inc}}\sin\omega\tau + W_{\text{ins}}\cos\omega\tau] \tag{17a}$$

and

$$W_{\text{out}} = \left[\eta_{hc} - \eta_{wic} + \xi L(\theta_c \sin I_i - \gamma_c \cos I_i) + \sum_{j=1}^{\infty} B_{ijc}\Phi_j(\xi)\right](-\omega\sin\omega\tau)$$
$$+ \left[\eta_{hs} + \xi L(\theta_s \sin I_i - \gamma_s \cos I_i) + \sum_{j=1}^{\infty} B_{ijs}\Phi_j(\xi)\right](\omega\cos\omega\tau)$$
$$= \omega[-W_{\text{outc}}\sin\omega\tau + W_{\text{outs}}\cos\omega\tau] \tag{17b}$$

In general, substitution of Eqs. (16) and (17) will not satisfy the linearized equations of motion Eqs. (14) for all τ. However, one can use Ritz's averaging technique, which involves multiplying both sides of each equation by $\cos\omega\tau$ and $\sin\omega\tau$ in turn and integrating over a period. The resulting algebraic equations may be solved to obtain the sine and cosine components of each generalized coordinate. The averaging integrations are straightforward for linear or simple velocity square damping terms in Eq. (14). Although the integrations of the form

$$\int_0^{2\pi} |\bar{W}_i| W_{\text{out}} \cos\omega\tau \, d(\omega\tau)$$

$$\int_0^{2\pi} |\bar{W}_i| W_{\text{in}} \cos\omega\tau \, d(\omega\tau)$$

etc. are more complicated, they can be evaluated in terms of elliptic integrals. For small inplane motions, however, these elliptic integrals reduce to simple algebraic expressions which were used in the actual computation.

III. Results and Discussion

Although the number of algebraic relations obtained from Eqs. (14) after the application of Ritz's averaging technique is infinite, in actual computation, only a finite number of them can be considered. Hence, only the first two terms in the expansion of v_i and w_i in Eq. (4a), which are likely to be the most significant ones, are taken into account. It was noted that this effectively includes the first four natural frequencies of the coupled system.

The degree of asymmetry in the motion of the legs is governed by the differences between the exciting wave displacements η_{wi} experienced by them. For the plane wave considered here, η_{w2} and η_{w3} are identical; but η_{w1} is different. It is clear from Eq. (15c) that for a given wave amplitude η_0, the difference between η_{w1} and η_{w2} is controlled by the product of $\exp(-2\pi H/\lambda)$ and $[\cos(\beta_1\xi - \delta_0) - \cos(\beta_2\xi - \delta_0)]$. If λ is reduced (i.e., ω is increased), the first term decreases while the second term becomes larger. Hence, when the excitation frequency ω is increased, $[\eta_{w1} - \eta_{w2}]$ moves towards a maximum but subsequently declines. The rotation λ also behaves similarly. However, θ is always zero due to the nature of the wave considered. The asymmetry of the motion also depends on δ_0, i.e., the distance of the crest of the wave from the vertical axis of symmetry z_0 of the system. If the z_0 axis is closer to the crest, there is comparatively less difference between the motions of the first and second (or third) legs. Typical amplitudes of wave excited response in the vicinity of the maxima are tabulated in Table 1 for a few practical combinations of the system parameters.

The inplane motion is absent for the type of single wave excitation examined in Table 1. However, it was retained in the computer program to tackle the complex waves where the water particles move horizontally as well.

It may be noticed from the table that unless H is very small, the asymmetry of the coupled motion may be ignored. Hence, for most practical situations, considerable simplification in the analysis and computer program can be achieved without substantially affecting the physics of the problem, by studying the dynamics of the system in which all the legs move identically. The influence of various parameters on the frequency response of the system is shown in Figs. 2-4. The dotted lines correspond to a uniform cylindrical cantilever while the solid lines describe the behaviour of the platform consisting of tubes having the same root diameter as the uniform one but with a taper ratio $\epsilon = 0.5$.

If the platform is to be used as a listening device, the hydrophones will be placed at the tips of the legs with the magnetometers at the central head. Hence, ideally, the plane containing the hydrophones should contain the central head. In Figs. 2-4, η_i, the absolute value of the difference between the amplitudes of the displacement of the head and that of the leg tip, has been plotted. As it is small compared to the head and buoy displacements, it has been magnified by a factor of 5 or 10.

Figure 2 shows the frequency response for two different Ω_b, which characterizes the cable stiffness. The peaks for η_b, η_h, and the absolute displacement of the tips (not shown) are moved to higher frequencies as Ω_b is increased. However, the frequency at which the maximum relative displacement

Table 1 Typical wave excited response in the vicinity of the maxima $\Omega_l = 2.0$, $\Omega_b = 0.2$, $r_{hl} = r_{bl} = 0.2$, $\bar{I}_x = \bar{I}_y = \bar{I}_z/2 = 0.005$, $\epsilon = 0.5$, $\eta_0 = 4.0$

						Tip displacements	
$\bar{H}$[a]	δ_0, deg	ω	η_b	η_h	γ, deg	$i=1$	$i=2,3$
3	45	0.70	3.25600	0.16417	0.59502	1.29850	1.30722
5	45	0.75	3.42039	0.06033	0.15679	0.68362	0.68643
10	45	0.30	2.78451	0.24376	0.22329	0.21793	0.21925
10	60	0.30	1.96878	0.15191	0.23957	0.13413	0.13539

[a] $\bar{H} = Hc/g(m_b + a_b)$.

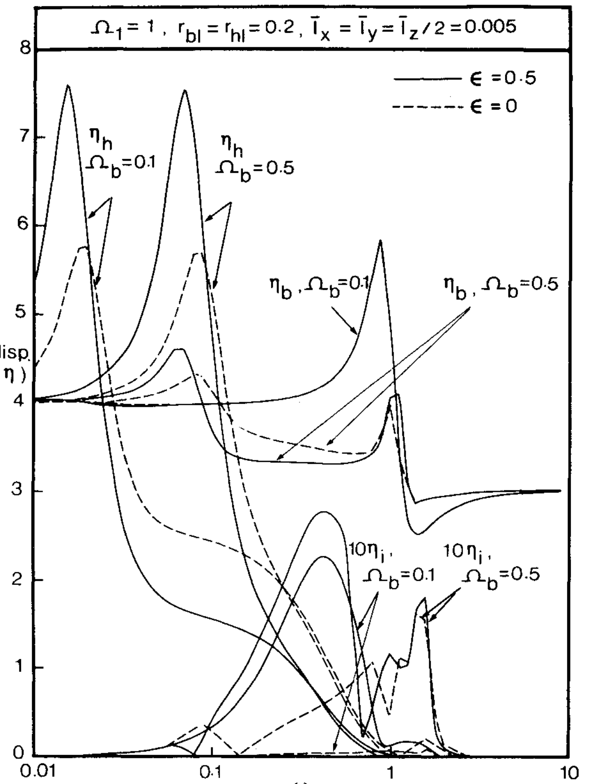

Fig. 2 Frequency response as affected by the cable stiffness.

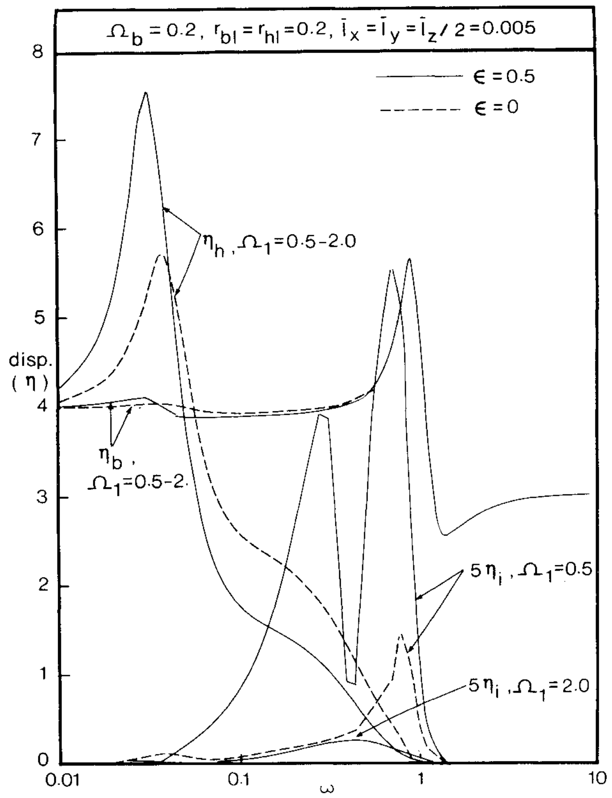

Fig. 3 Effect of the fundamental frequency of each leg on the system response.

between the tips and the head occurs, remains more or less unchanged. The maximum values of η_b, η_h, and η_i are reduced, unaltered and increased, respectively, for a larger Ω_b. Making the tubes tapered increases the peak values of the displacements but helps the system performance in certain range of the forcing frequency.

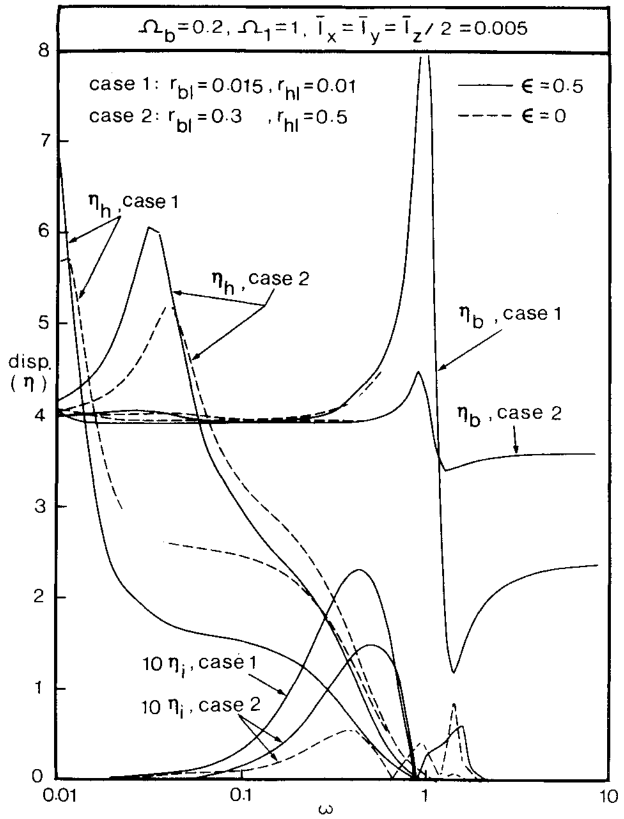

Fig. 4 Influence of the buoy and central head inertias on the system response.

The influence of the stiffness of the legs on the wave excited motion is plotted in Fig. 3. The amplitudes of the buoy and head are unaffected in the range of $\Omega_l\,(=\lambda_l^2\Omega)$ considered; however, the relative motion between the tip and the root of the tubes decreases for a larger Ω_l. For $\Omega_l = 2.0$ and $\epsilon = 0$, η_i is negligible and hence not shown in the plot.

The inertia parameter r_{hl} may be increased by a heavier central head. But in that case, a large portion of the buoy remains under water, causing a higher added mass of the buoy during motion and increasing r_{bl}. The effect of a heavier head is to reduce the maximum values of η_h, η_b, and η_i of a tapered tube, but to increase η_i of a uniform tube (Fig. 4).

IV. Concluding Remarks

The important conclusions based on the analysis can be summarized as follows:

1) The asymmetry in the motion of the legs and the rotation of the platform are negligible for most practical situations.

2) The displacement of the tip of a leg relative to the central head can be reduced by using a cable with smaller stiffness and stiffer legs.

3) Although making the legs tapered increases the peak values of η_b, η_h, and η_i, it may improve the system response in certain frequency ranges.

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References

[1] Leonard, R. W., "State-of-the-Art in Inflatable Shell Research," *Journal of Engineering Mechanics,* Vol. 100-EM1, Feb. 1974, pp. 17-25.

[2] Kornecki, A., "A Note on Beam-Type Vibrations of Circular Cylindrical Shells," *Journal of Sound and Vibration,* Vol. 14, Jan. 1971, pp. 1-6.

[3]Goldenveizer, A. L., *Theory of Elastic Thin Shells*, Pergamon Press, N.Y., 1961, p. 230.

[4]Housner, G. W. and Keightley, W. O., "Vibrations of Linearly Tapered Cantilever Beams," *Journal of Engineering Mechanics*, Vol. 88-EM2, April 1962, pp. 95-123.

[5]Gaines, J. H. and Volterra, E., "Transverse Vibrations of Cantilever Bars of Variable Cross-Section," *Journal of the Acoustical Society of America*, Vol. 39, April 1966, pp. 674-679.

[6]Modi, V. J. and Poon, D. T., "Dynamics of Neutrally Buoyant Inflated Viscoelastic Tapered Cantilevers Used in Underwater Applications," *ASME Design Engineering Technical Conference*, Chicago, Sept. 1977, Paper 77-Det-92; also *Journal of Machine Design*, Vol. 100, April 1978, pp. 337-346.

[7]Modi, V. J. and Misra, A. K., "Vertical Motion of a Buoy-Cable-Array System Used in Submarine Detection," *Journal of Hydronautics*, Vol. 10, July 1976, pp. 74-81.

[8]Meirovitch, L., *Analytical Methods in Vibrations*, MacMillan, N.Y., 1967, p. 163.

[9]Keulegan, G. H. and Carpenter, L. H., "Forces on Cylinders and Plates in an Oscillating Fluid," *Journal of Research of the National Bureau of Standards*, Vol. 60, May 1958, pp. 423-440.

[10]Wiegel, R. L., *Oceanographical Engineering*, Prentice-Hall, Englewood Cliffs, N.J., 1964, pp. 11-21.